

REGULAR STALAGMITES: THE THEORY BEHIND THEIR SHAPE

TEORETIJA RASTI PRAVILNIH STALAGMITOV

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Abstract

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Wolfgang Dreybrodt & Douchko Romanov: Regular stalagmites: The theory behind their shape

Under growth conditions constant in time stalagmites grow into an equilibrium shape, which is established, when all points of its surface are shifting by the same vertical distance $W_0 = 1.2 \cdot 10^9 \cdot \alpha(c_{in} - c_{eq}) \cdot \Delta t$ (cm/year) during a time interval Δt (years). Thereby $\alpha(c_{in} - c_{eq})$ is the precipitation rate in molcm⁻²s⁻¹, c_{eq} is the calcium concentration of the supersaturated solution dripping to the apex of the stalagmite, and c_{in} its equilibrium concentration with respect to calcite and the p_{CO_2} in the cave atmosphere.

From these ingredients a numerical model of stalagmite growth into an equilibrium shape is presented. In this model one assumes idealistically that the water dripping to the apex flows continuously down the stalagmite, spreading out radially. By simple mass balance one finds that the equilibrium radius is $R_{eq} = \sqrt{V/\pi\alpha\tau}$, where $V = 0.1$ cm³ is the volume of a drop and τ the drip interval. Furthermore numerical modeling reproduces the vertical shifting of the stalagmite's equilibrium shape. Finally an interesting similarity rule is found. If one scales two stalagmites of differing R_{eq} to the same size and chooses their growth axes as common axis and their apexes as common origin, both show identical shapes. In other words regular stalagmites are similar geometrically. This similarity rule is verified by digitizing the shapes of various natural stalagmites with diameters between 5 cm and 20 m. Within small natural variations, the rescaled shapes are identical and close to the shape of the numerical model.

Keywords: stalagmites, morphology, growth rates, scaling law

Izvleček

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Wolfgang Dreybrodt & Douchko Romanov: Teorija rasti pravilnih stalagmitov

V stacionarnih pogojih rasti stalagmiti zrastejo v ravnotežno obliko. Todosežejo, ko se vse točke na površini v določenem časovnem intervalu Δt premaknejo za enako vertikalno razdaljo $W_0 = 1.2 \cdot 10^9 \cdot \alpha(c_{in} - c_{eq}) \cdot \Delta t$ (cm/year), pri čemer je $\alpha(c_{in} - c_{eq})$ hitrost odlaganja sige v molcm⁻²s⁻¹, c_{in} je koncentracija kalcija v raztopini, ki kaplja na teme kapnika, c_{eq} ravnotežna koncentracija kalcija glede na kalcit, p_{CO_2} pa je delni tlak ogljikovega dioksida v jamski atmosferi. Iz teh sestavin je narejen model razvoja stalagmita v ravnotežno obliko. V modelu predpostavimo, da voda odteka iz temena stalagmita enakomerno v radialni smeri. Iz ohranitve mase dobimo ravnotežni radij stalagmita $R_{eq} = \sqrt{V/\pi\alpha\tau}$, pri čemer je $V = 0.1$ cm³ volumen vodnih kapljic in τ interval kapanja. Numerični model napove vertikalno premikanje ravnotežne oblike. Z ustrezno transformacijo koordinat se nazorno izkaže, da so pravilni stalagmiti geometrijsko identični. To v članku prikažemo na izbranih primerih naravnih stalagmitov premerov med 5 cm in 20 m. V okviru majnih odstopanj so umerjene oblike stalagmitov identične z napovedi numeričnega modela.

Ključne besede: stalagmit, morfologija, hitrost rasti, skalna invarianca.

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INTRODUCTION

Stalagmites are ubiquitous in many caves. They grow in a variety of shapes and sizes. Their diameters range from a few centimetres up to several meters. Fig. 1 shows a transect of a small regular stalagmite with a diameter of 4.5 cm (Soubies *et al.* 2005). The largest stalagmite in the world is reported from Cueva San Martin Inferno in Cuba. This giant stalagmite exhibits a height of about 70 m and 20 m diameter. The photo of this speleothem <http://www.goodearthgraphics.com/virtcave/largest.htm> is shown in Fig. 2. Both samples, although extremely different in size at a first glance look quite similar. Both shapes can be crudely described as almost regular cylinders with a half sphere at their top. Such regular stalagmites sometimes are called candle stalagmites.



Fig. 1: Transect of a small regular stalagmite (Soubies *et al.* 2005) with a diameter of 4.5 cm. The laminations show growth surfaces. The red lines show the distance between various points of two distinct growth surfaces. They are all of the same length.

Franke (1965) was the first who recognized the principle of their growth. He proposed the following mechanism assuming growth conditions constant in time. A drop of solution, supersaturated with respect to calcite falling to the ground spreads out radially when it impinges at the surface. Precipitation rates are maximal at the center and decrease with distance outwards. This way a thin layer of precipitated calcite is created. It has a lenticular shape, thick at the center and thinning out at the edges. The next

layer is build in the same way on the top of the previous one and a vertical candle like shaped stalagmite grows upwards. After a height of about its diameter its shape on the



Fig. 2: The world's largest stalagmite in Cueva San Inferno, Cuba. Its height is 70 m, its diameter 20 m.

top becomes stable and also the diameter of the stalagmite does not change anymore. In other words the growth rates in the vertical direction are constant everywhere and the shape of the stalagmite is shifted vertically in its growth direction. This is depicted in Fig. 3. Franke suggested this principle from observation on the growth layers of transected stalagmites, such ones as shown in Fig. 1. Dreybrodt (1988,1999) and Romanov *et al.* (2008a) derived this growth principle from three simple generally valid assumptions by use of numerical modelling, but also by strict mathematical proof. These are:

- 1) The point of drop impact and the drip interval are constant in time. This must be true also for temperature.
- 2) Growth by deposition of calcite is directed normal to the actual surface of the stalagmite.

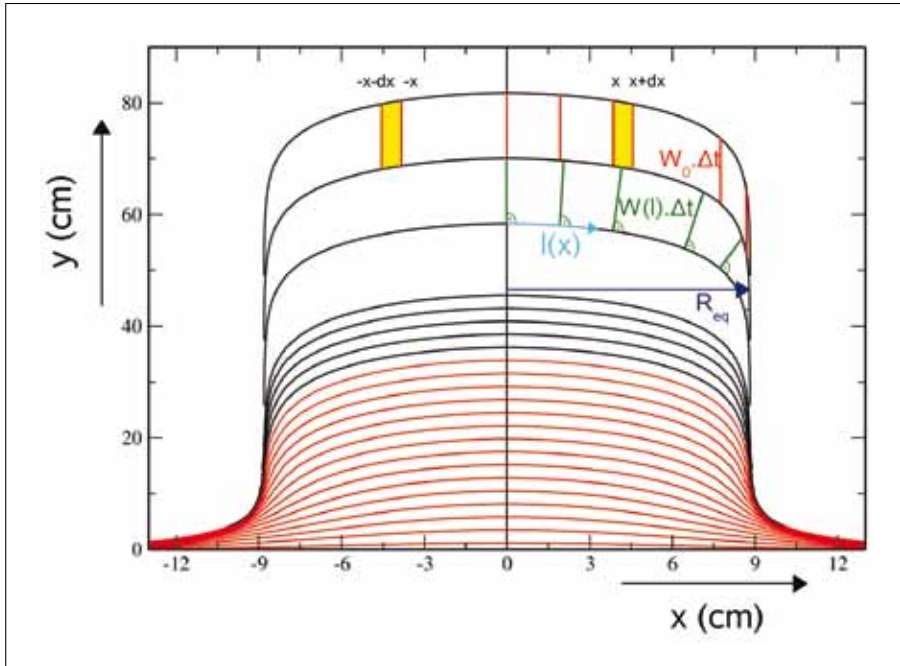


Fig. 3: Principle of stalagmite growth: The green lines depict the growth perpendicular to a growing surface during the time span Δt . They decrease with distance l along the growing surface. The red curves show the shapes of the stalagmite when it starts to grow from a plane surface. Finally an equilibrium shape with radius R_{eq} is attained (black curves). The vertical distance between to growth surfaces is shown by the red arrows with length $W_0 \cdot \Delta t$, they are all of equal length. See also Fig. 1. The volume between the two utmost layers is composed of rings with radius x , width Δx and height $W_0 \cdot \Delta t$. Their cross section is a parallelogram with sides Δx and $W_0 \cdot \Delta t$. See text.

3) Growth rates $W(l)$ decrease monotonically with distance l along the actual surface and become zero at some large but finite distance l_0 .

From these assumptions one can derive also a similarity rule depicted in Fig. 4: Rescaling the shape of a regular stalagmite into dimensionless coordinates $\tilde{x} = x/R_{eq}$ and $\tilde{y} = y/R_{eq}$, choosing their growth axes as common axis, and their apexes as origin $\tilde{x} = 0, \tilde{y} = 0$ gives identical shapes, independent of the diameter R_{eq} of the sample. (Romanov *et al.*, 2008a)

In this paper we will first discuss the basic features of stalagmite growth to understand this similarity rule. In the second step we use rescaling of real regular stalagmites from diameters in the range of several centimeters up to about 20 m to show its validity.

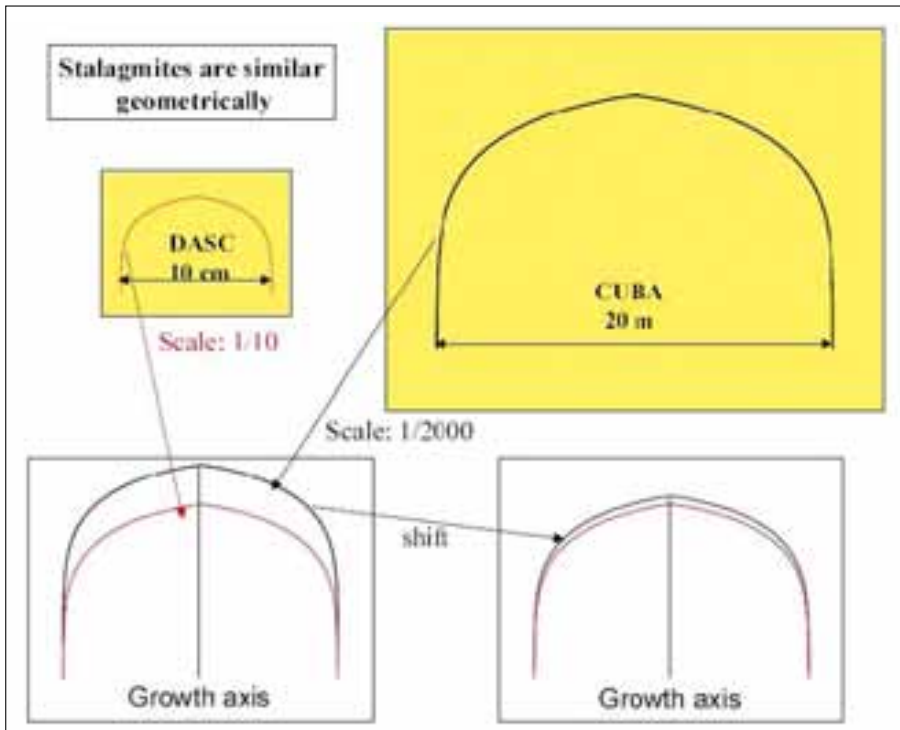


Fig. 4: Schematical representation of the geometrical similarity of stalagmites: A small stalagmite with 10 cm diameter is scaled by a factor of 1/10. A second one of 2000 cm diameter is scaled by a factor of 1/2000. In the new common coordinate system they are plotted with common growth axis (a). Then the red line is shifted vertically until it merges with the black line (b).

BASIC PRINCIPLES OF STALAGMITE GROWTH

The most important principle stated already, can be formulated as follows: Under constant growth conditions in time the top of a stalagmite attains an equilibrium shape, which during further growth is shifted up vertically. The part of the stalagmite below is a vertical cylinder with constant equilibrium radius R_{eq} . This is illustrated in Fig. 3. This principle has been proven by Dreybrodt (1988) and Romanov *et al.* (2008a) as already mentioned in the introduction. We will not repeat this proof here, but we will derive some important consequences from it.

Fig. 3 shows two growth surfaces of a stalagmite in equilibrium, which are separated by some distance $W_0 \cdot \Delta t$ (cm/year) is the growth rate at the apex and Δt (year) the time distance when the two layers originated. The total volume V between these two layers is composed of circular rings with radius x and width Δx , all with equal height $W_0 \cdot \Delta t$. Summing over all of them yields

$$V = W_0 \cdot \Delta t \int_0^{R_{eq}} 2\pi x dx = W_0 \pi R_{eq}^2 \cdot \Delta t \quad (1)$$

W_0 is the growth rate at the apex in cm/year.

Precipitation of calcite from thin films of supersaturated solutions, as they rest on stalagmites has been investigated theoretically and experimentally by Buhmann and Dreybrodt (1985) and Dreybrodt *et al.* (1996). These findings have been verified by measuring precipitation rates on stalagmites in relation to the chemical composition of the drip water (Baker and Smart, 1995, Baker *et al.*, 1998, Genty *et al.* 2001). The precipitation rates R are given by

$$R = \alpha(c - c_{eq}) \text{ mol cm}^{-2}\text{s}^{-1} \quad (2)$$

where α [cm/s] is the rate constant, c [mol/cm³] is the actual calcium concentration in the solution, and c_{eq} its equilibrium concentration with respect to calcite and the partial pressure p_{CO_2} of CO_2 in the cave atmosphere. α depends on temperature. Values of α in the range between 0°C and 30°C are reported by Baker *et al.* (1998). These can be fitted by the relation

$$\alpha = (0.52 + 0.04 \cdot T_c + 0.004 \cdot T_c^2) \cdot 10^{-5} \text{ cm/s} \quad (3)$$

where T_c is the temperature in centigrade (Romanov *et al.* 2008b). From this one gets a increase of a factor of ten with increasing temperature from 0°C ($\alpha = 5.2 \cdot 10^{-6}$ cm/s) to 30°C ($\alpha = 5.3 \cdot 10^{-5}$ cm/s).

To convert the precipitation rates R (mol/(cm² s)) into a growth rate W in cm/year we consider the volume of calcite deposited to 1 cm² in 1s: This is given by

$$V_s = R \cdot \frac{M}{\rho} = R \cdot \frac{100}{2.7} = 37.07 \cdot R \quad (4)$$

M is the molecular weight of $CaCO_3$, $M = 100$ g/mol and $\rho = 2.7 \frac{g}{cm^3}$ the specific weight of calcite. To convert this into growth of cm per year one has to multiply by the number N_s of seconds in one year and finds $1 \text{ mol cm}^{-2}\text{s}^{-1} = 1.168 \cdot 10^9 \text{ cm/year}$.

Therefore

$$W = R \cdot \frac{M}{\rho} \cdot N_s = 1.168 \cdot 10^9 \alpha (c - c_{eq}) \text{ cm/year} \quad (5)$$

Now the volume V_d deposited to the stalagmite in 1 year ($\Delta t = 1$) must be equal to the volume V_L of the amount of calcite lost during this time from the water, which enters with concentration c_{in} at the apex and leaves the stalagmite with equilibrium concentration c_{eq} .

$$V_L = \frac{V_{drop}}{\tau} \cdot (c_{in} - c_{eq}) \cdot \frac{M}{\rho} \cdot N_s = \frac{V_{drop}}{\tau} \cdot (c_{in} - c_{eq}) \cdot 1.168 \cdot 10^9 \quad (6)$$

$$V_d = W_0 \pi R_{eq}^2 = 1.168 \cdot 10^9 \alpha \cdot (c_{in} - c_{eq}) \cdot \pi R_{eq}^2$$

V_{drop} is the volume of the drop and τ the drip interval.

Therefore we find by equating $V_L = V_d$

$$R_{eq} = \sqrt{\frac{V_{drop}}{\pi \alpha \tau}} \quad (7)$$

This result relates in a simple way the equilibrium radius to the kinetic constant α and the drip interval τ . Note that the radius does not depend on the initial supersaturation ($c_{in} - c_{eq}$).

As an example one finds with $\alpha = 1.3 \cdot 10^{-5}$ cm/s, corresponding to a temperature of 10°C, a drop volume $V_{drop} = 0.1 \text{ cm}^3$, and $\tau = 30$ s the equilibrium radius $R_{eq} = 9$ cm, a very realistic number.

This equation also tells us that changes in drip interval τ and in kinetic constant α cause changes of the equilibrium radius and therefore also changes to the shape of the stalagmite. Increase in the drip interval τ or in α

result in smaller diameters and cone like stalagmites can result. On the other hand decreasing τ or α lead to an increase of R_{eq} and club-like shapes arise. This has been shown by numerical simulation first by Dreybrodt (1988, 1999). Kaufmann (2002) and Kaufmann and Dreybrodt (2003) have taken climatic data from ice cores and tried

to find out, how these were imprinted into the shape of stalagmites. They showed that analyzing the shapes of irregularly grown stalagmites could give some additional information to the interpretation of climate proxies from oxygen and carbon isotopes.

A MODEL OF THE EQUILIBRIUM SHAPE AND THE SCALING RULE

In our approach we assume that the water supplied to the apex spreads out to a water film, which expands radially flowing down the stalagmite in laminar flow. This way the stalagmite is covered by a water film of thickness δ with radial symmetry as shown in Fig. 5.

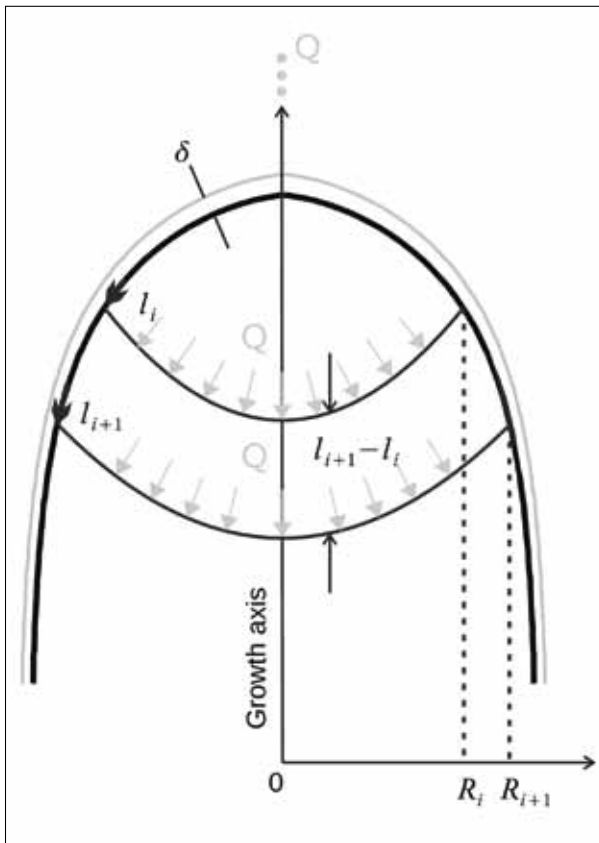


Fig. 5: A water film with thickness δ covers the stalagmite. An annulus of this film between distance l_i and l_{i+1} is shown. The water supply Q flows radially from the apex. At distance l_i along the growth surface the water enters into the annulus with concentration c_i and leaves it at distance l_{i+1} with concentration c_{i+1} .

We now consider an annulus between the points i and $i+1$ at the surface with radial distance R_i and R_{i+1} re-

spectively. The distance between these two points along the surface is $l_{i+1} - l_i = \Delta l_i$.

Mass balance requires that the amount of calcite deposited to the area of the annulus between points i and $i+1$ must be equal to the loss of CaCO_3 in the solution when passing over the annulus.

The mass of calcite lost per second from the solution covering the surface of the annulus is

$$M_s = Q(c_i - c_{i+1}) \quad (8)$$

because the total flow $Q = V_{\text{drop}}/\tau$ enters the annulus at R_i with concentration c_i and leaves it at R_{i+1} with concentration c_{i+1} . M_s must be equal to the mass M_p of calcite precipitated per second to the surface $S_a = 2\pi \cdot \Delta l_i \cdot R_i$ of the annulus. Note that M_s and M_p in this context are in mol/s.

$$M_p = 2\pi \cdot \Delta l_i \cdot R_i \cdot \alpha \cdot (c_i - c_{eq}) \quad (9)$$

Equating $M_p = M_s$ one obtains

$$c_{i+1} = c_i - 2\pi \cdot \alpha \cdot (c_i - c_{eq}) \cdot R_i \frac{\Delta l_i}{Q} \quad (10)$$

From this the deposition rates W_{i+1} can be calculated by subtracting c_{eq} from both sides of the equation and then multiplying both sides by α

$$W_{i+1} = W_i \left(1 - 2\pi \cdot \alpha \cdot R_i \frac{\Delta l_i}{Q}\right) \quad (11)$$

Remembering that $R_{eq} = \sqrt{\frac{V_{\text{drop}}}{\pi \alpha \tau}} = \sqrt{\frac{Q}{\alpha \pi}}$ eqn. 11 can be rewritten as

$$W_{i+1} = W_i \cdot \left(1 - 2 \cdot \frac{R_i \Delta l_i}{R_{eq}^2}\right) \quad (12)$$

This recursive equation shows growth rates declining with distance l_i . Rewriting eqn.11 as

$$\frac{W_{i+1} - W_i}{W_i} = \frac{W(l_{i+1}) - W(l_i)}{W(l_i)} = 2 \cdot \frac{R_i \Delta l_i}{R_{eq}^2} \quad (13)$$

In the limit of very small $\Delta l_i = dl$ we obtain

$$\frac{dW(l)}{W(l)} = \frac{2Rdl}{R_{eq}^2} \geq \frac{2dl}{R_{eq}^2} \quad (14)$$

because $l(R) \geq R$.

Integrating eqn. 14 yields

$$W(l) \leq W_0 \cdot \exp\left(-\frac{l^2}{R_{eq}^2}\right) \quad (15)$$

Since the Gauss-function is monotonically decreasing with distance l this must be true also for $W(l)$.

Equation 11, which contains only defined parameters and the somewhat idealistic assumption of continuous flow spreading out radially can be used for numerical modelling. This has been performed as FLOW-model by Romanov *et al.* (2008). They employed the following procedure. The actual surface of the stalagmite at time t is approximated by the polygon P_1 to P'_n where P_1 is the point of drip impact. The surface at later time $t+\Delta t$ is obtained by drawing vertical lines with lengths $W(l_i) \cdot \Delta t$ from points i . This way the three basic assumptions on stalagmite growth are fulfilled and the new surface is approximated by the polygon P'_1 to P'_n as shown in Fig. 6.

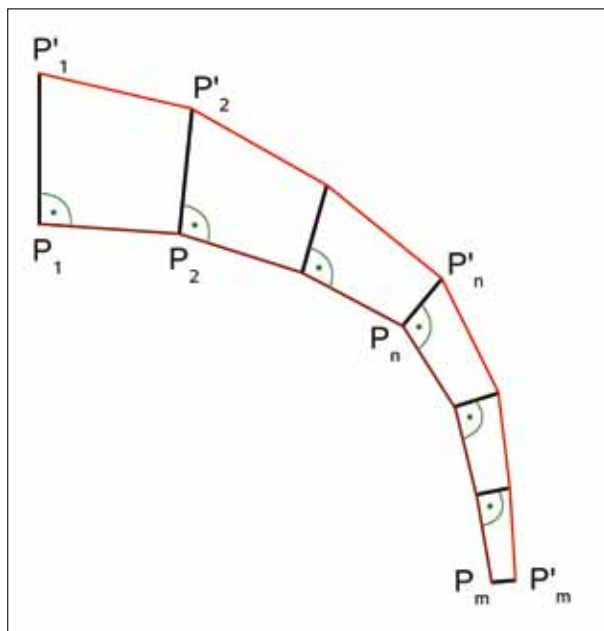


Fig. 6: Consecutive construction of a new growth surface $P'_1 P'_n$ from an actual surface $P_1 P_n$. The distances $P_n P'_n$ depend on the lengths along the actual surface $P_1 P_n$. They decrease with n . P'_1 are the points of drip impact (see text).

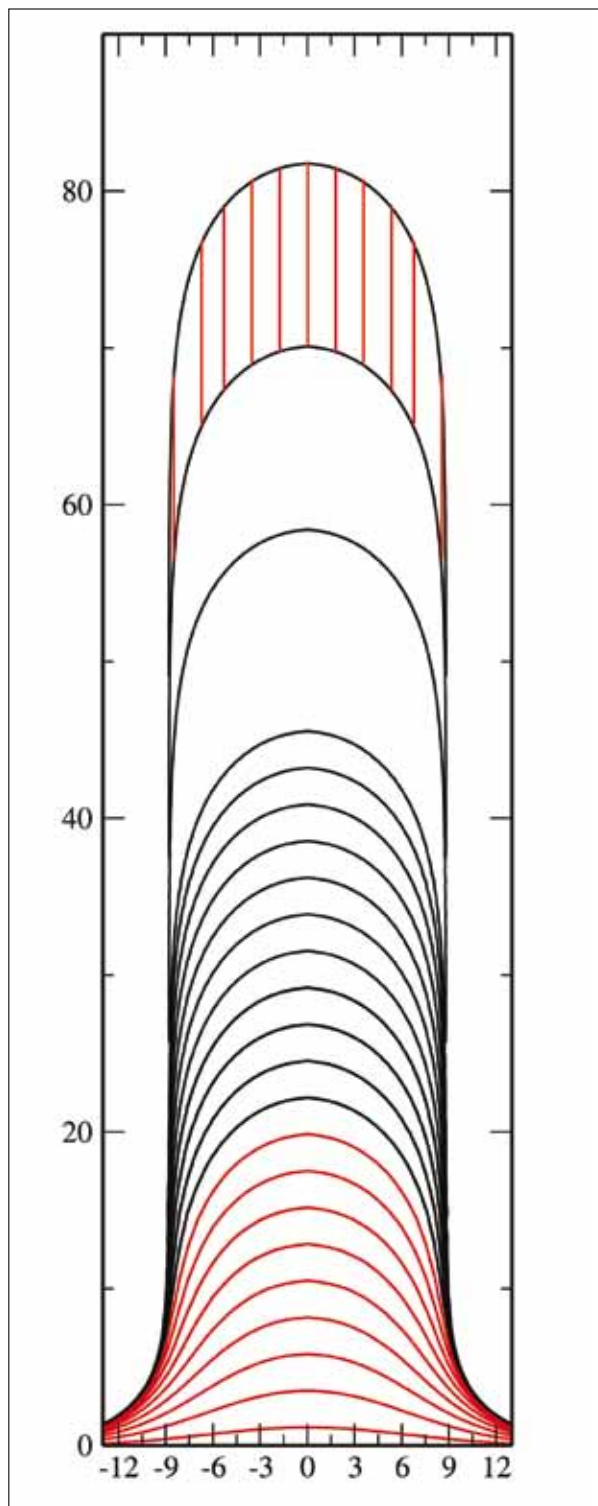


Fig. 7: Computer simulation of a stalagmite by the FLOW-model: $V_{drop} = 0.1 \text{ cm}^3$, $\tau = 30 \text{ s}$, $\alpha = 1.3 \cdot 10^{-5} \text{ cm/s}$ ($T = 10^\circ \text{C}$). The red lines show the initial state of growth until an equilibrium shape is attained (black curves). The growth surfaces are shown every 200 years in the lower part and every 1000 years in the upper part. The vertical lines between the two utmost surfaces are all of equal length $W_0 \cdot \Delta t$. $\Delta t = 1000 \text{ years}$

Here we do not give technical details but we represent the results of the FLOW- model. These are shown in Fig.7. It depicts growth layers of a stalagmite growing on a plane surface. Note that the x and y axes have identical scales. Therefore the picture presents the natural undistorted shape. Clearly the shape of the stalagmite changes in its initial stage of growth (black lines) until an equilibrium shape is attained (red line) from then on this shape is maintained for further growth and the vertical distance between any two equilibrium growth layers remains constant on all points of the surface. The vertical red lines, all of equal lengths between the two upper surfaces indicate this. A further point to be noted is that the equilibrium shape is attained after a growth height close to its diameter.

The similarity rule can be obtained from eqn.12. By introducing dimensionless coordinates $\tilde{x} = x/R_{eq}$ and $\tilde{y} = y/R_{eq}$ one finds

$$W_{i+1} = W_i \cdot (1 - 2 \cdot \tilde{R}_i \cdot \Delta \tilde{l}_i) \quad (16)$$

This, however, means that any regular stalagmite scaled by the factor $1/R_{eq}$, into this new coordinate system must exhibit a unique shape independent on its former dimensions.

This proves the scaling rule depicted in Fig. 4: Scale the stalagmites in both coordinates by the factor $1/R_{eq}$ and chose their growth axis as y-coordinate and their apex as origin. Then all regular stalagmites have the same shape.

VERIFICATION OF THE SCALING RULE ON NATURAL STALAGMITES

To prove this by observation one has to select regular stalagmites, which show vertical walls about 1 diameter below their apex. Two extreme examples are shown in Figs. 1 and 2.

We have selected six transections of stalagmites, which are exhibited in the Deutsche Archiv für Sinterchronologie (DASC) in the Schillat-cave, located in a limestone quarry close to Hessisch-Oldendorf, Germa-

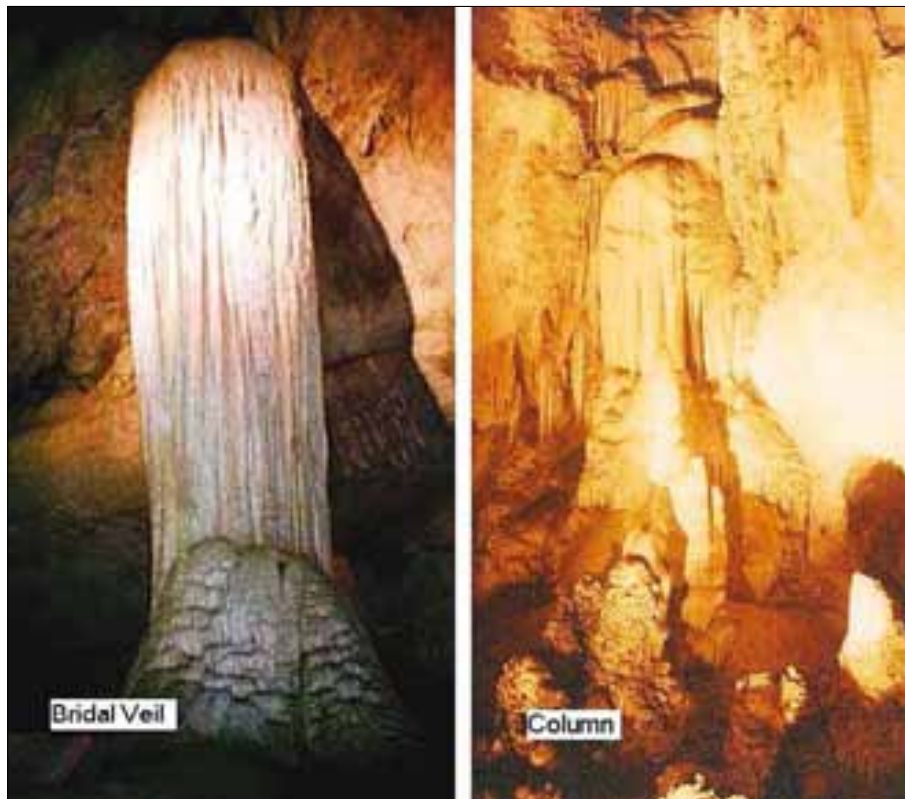


Fig. 9: Two large stalagmites: Column from Kings Palace, Carlsbad Caverns, New Mexico, USA and the Bridal Veil, Lost world Caverns, West Virginia, USA.

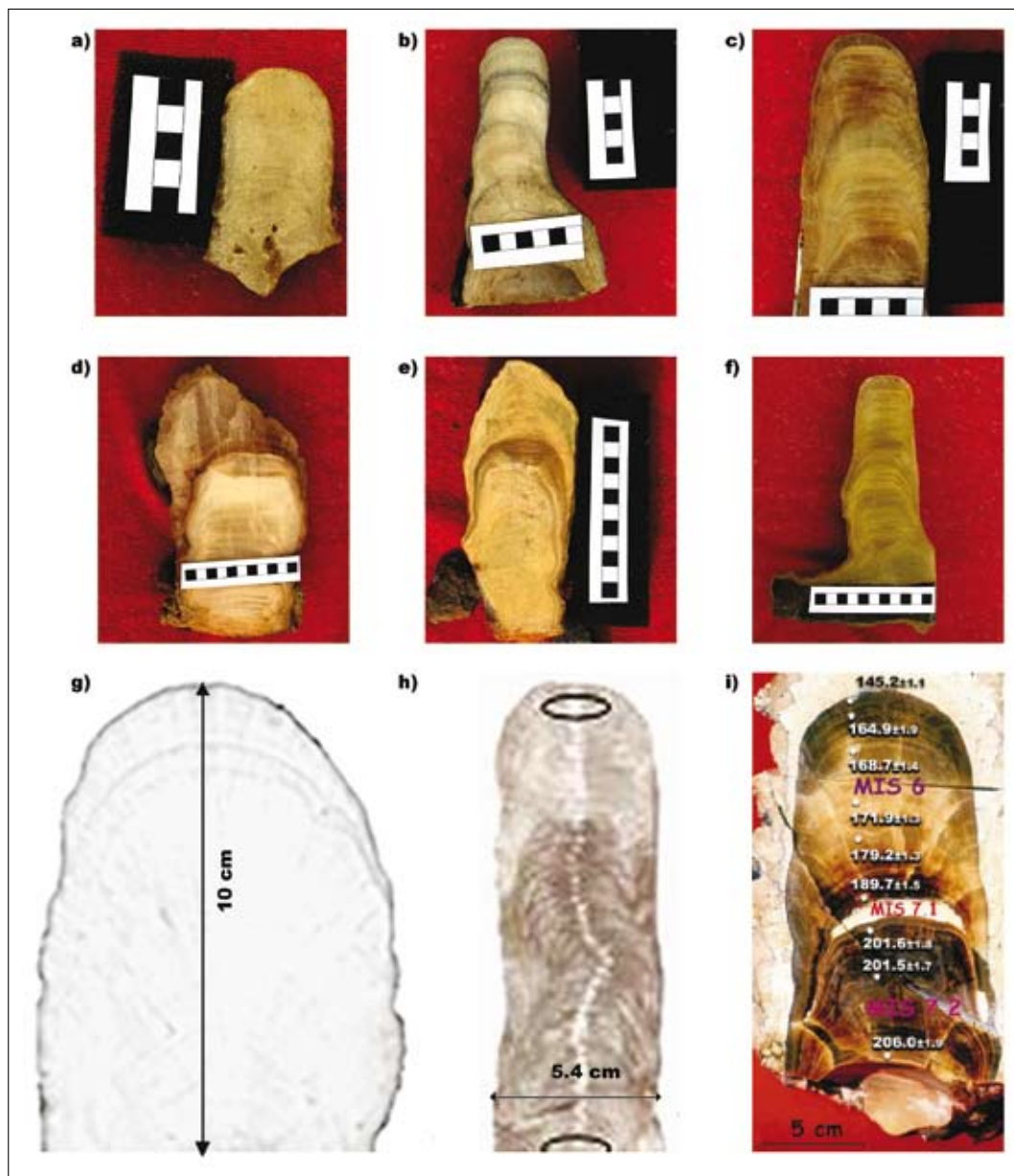


Fig. 8: Regular small stalagmites from the Deutsche Archiv for Sinterchronologie (DASC) used to test the similarity rule. Regular stalagmites taken from the literature are also shown (see text). a) dasc1, b) dasc2, c) dasc3, d) dasc4, e) dasc9, f) Frisia, g) Genty, h) Bard. The black and white squares at the base of stalagmites are 1 cm x 1 cm.

ny. These stalagmites exhibit diameters between 4 cm to 8 cm. They are shown in Fig. 8.

Furthermore four stalagmites have been taken from the literature: Bard et al. (2002), Fig. 1, Frisia et al. (2006), Fig. 2, both with diameters of 10 cm, Genty et al. (2006), Fig. 4, with a diameter of 5 cm, and Soubies et al. (2005), Fig. 2a with 4.5 cm diameter. These are also shown in Fig. 8. We have digitized suitable growth layers of the surface of those stalagmites and rescaled them. We have also performed a search of regular large stalagmites

in the Internet and found the world largest regular stalagmite with a diameter of 20 m shown in Fig. 2. Other ones have been found on the from Carlsbad Caverns, New Mexico, USA, a column with 4 m diameter (<http://www.fingerlakesbmw.org/visual/az/cccolumn1.jpg>) and finally from Lost World Caverns, West Virginia, USA, the Bridal Veil with 2 m diameter, (<http://www.wonderfulwv.com/archives/june01/fea1.cfm>). Pictures of these two stalagmites are shown in Fig. 9.

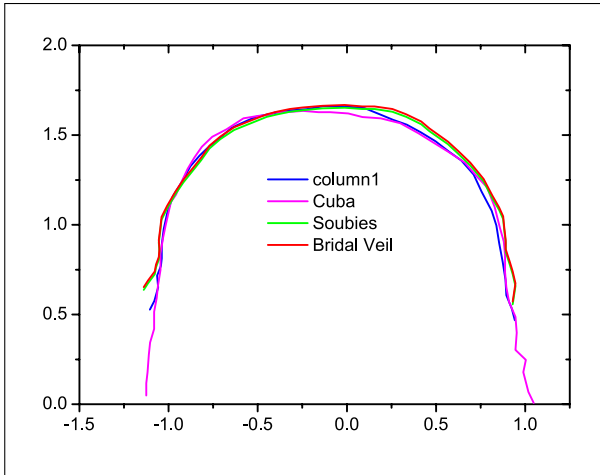


Fig. 10: Rescaled shapes of three large stalagmites and one small stalagmite. See color legend.

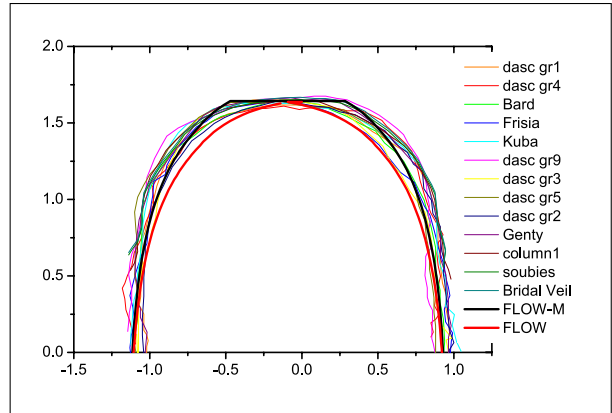


Fig. 12: Rescaled shapes of all stalagmites compared to the shape obtained from the FLOW-model and a modified FLOW-model FLOW-M.

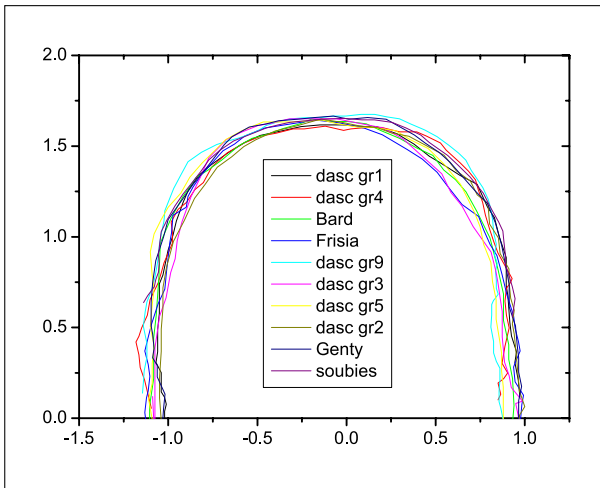


Fig. 11: Rescaled shapes of the stalagmites from Fig. 8. See color legend.

To compare the shapes we have rescaled all stalagmites to the same radius $\tilde{R}_{eq} = 1$.

In Fig. 10 the rescaled shapes of the stalagmite from Cuba, Column, and the Bridal Veil are shown. Within some natural variation all these shapes are almost the same. We have also added the smallest stalagmite with 4.5 cm from the paper of Soubies *et al.* (2005), (see Fig. 1). Its shape fits nicely with those of the large stalagmites.

Fig.11 compares the shapes of all small stalagmites from Fig. 1 and Fig. 8, which also prove the similarity rule. Finally, in Fig. 12 we have plotted the complete selection of our samples and compare it to the theoretically obtained rescaled shape of the FLOW-model depicted in Fig. 7.

Although not perfect the theoretical shape is quite close to the natural shapes.

DISCUSSION AND CONCLUSION

We have shown that the shapes of regular stalagmites are similar from very small equilibrium diameters of several centimetres over three orders of magnitude up to 20 m. This is a rather unexpected result because the hydrodynamic flow conditions are extremely different. Small stalagmites up to diameters of several 10 cm are fed by drops with drip intervals of about 10 s for a diameter of 30 cm and 200 s for a diameter of about 5 cm. To obtain a diameter of 20 m a water supply of 40 cm³/s or 400 drops/s are necessary. Whenever a drop impinges to the surface of a stalagmite splashing will occur and the drop

or the continuous flow of water will cover the central part of the stalagmite's surface by water film, which acts like a reservoir from which outward flow is supplied. In this water film due to mixing, when a new drop impinges, a constant calcium concentration will be maintained and the growth rate is constant within the radius where this water film covers the stalagmite.

To model the shape of this stalagmite requires $W_i = W_0$ for all points i with $R_i \leq R_c$. For $R_i > R_c = 0.33 \cdot R_{eq}$ the recursive equation 11 is valid.

The comparison of the equilibrium shapes is shown in Fig. 12. The new shape (black fat line) is exactly what one obtains by cutting the upper part of the idealised stalagmite (obtained from the FLOW – model). The modified model FLOW-M comes closer to the real shapes although it is not perfect. The FLOW- model exhibits a sharper tip.

Concluding we can state that although hydrodynamics are different for small and large stalagmites, they all obey a similarity law, which indicates that the detailed growth mechanisms must be very similar. It may be a topic of future research to incorporate the hydrodynamics including surface tension of the water flowing away from the apex into further more realistic growth models.

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